**Shukla**

Motivates paper by assertion that in nanodevices there can be great variation of conductance properties even with same macroscopic parameters (like number of impurities, external fields, length, boundary conditions). It is of interest to study this variation. The conductance can be related to the eigenvalues of the transmission matrix, or rather of tt†, and so it is aimed to determine probability distribution of these eigenvalues.

The approach to developing such a scaling equation starts naturally from the transfer matrix. One can hopefully develop an equation that describes the development of this matrix from which the transmission eigenvalues can be obtained. It had been assumed that the entire transfer matrix itself could be described by a probability distribution independent of the particular details of the Hamiltonian of the system. This is how random matrix theory was applied to the problem. It was assumed that the transfer matrix was randomly/uniformly distributed (or rather it maximized the information entropy within this ensemble) within its symmetry class (the symmetry class being the set of matrices that satisfied the symmetries of the Hamiltonian: i.e. flux conservation for sure, then perhaps TRS, SRS, etc.), subject to the constraint that it produced the correct average eigenvalue (λ) density. One could then, using the polar representation, determine how the eigenvalues λ, themselves, were distributed. And hence one could obtain the probability distribution of the conductance. This attempt was partially fruitful as it, by design, correctly predicted the UCF, but it got the value slightly wrong.

Next attempt to develop a scaling equation for P(M) only assumed that the infinitesimal slice was distributed in max entropy way according to its class, subject to the requirement that for that slice generate the correct infinitesimal conductance, via correct transfer coefficient, i.e.,



One obtained the result:



Scaling equation so obtained was DMPK equation and it correctly predicted variance coefficient, etc., for Q1D systems in the weak scattering limit. Note that since the slice’s u, υ’s were distributed isotropically, it could be shown that the overall u’s, υ’s were too. And so that is how one could do the averages over u and υ during the development of the DMPK equation. Of course Mello showed that we need not assume the u,υ’s are isotropically distributed, but merely that the channels are equivalent, in that they have the same m.f.p. I’m not sure what this would entail regarding their distribution.

But certainly, we cannot assume isotropy in any event, if we are to generalize the problem. Isotropy presumes that the flux is already evenly distributed throughout all channels, which means the width is much smaller than the localization length. The way Shukla puts it, he says that the correlations in the building block are only those of the measure dμ(M), and this measure arose strictly through the requirement of current conservation. But this is too vague.

**1. Random matrix model**

Seems like he’s not committing to a microscopic model per seʹ, but is alleging that the entire matrix is a Gaussian statistic. So in other words, p­L(M) = whatever, which maximizes entropy subject to symmetry constraints, average, correlation constraints.



Well we have:



He writes this a little differently however. For short he writes Mμν as Mμ. And Aμν = 2aμ, and Bμν;μ´ν´ = bμμ´. So now we have:



Normalization requires:



Note that the measure dμ(M) will contain reference (probably) to complex elements Mμν\*, as the measures dμ(u) and dμ(υ) do. The <M> constraint requires:



And the <MM´> costraint requires:



We can write these constraints in the following way. For instance consider the <M> constraint.



Doing a similar calculation for the correlation term and we have:



He says that a Gaussian function is reasonable to expect since the sample will contain many independent scatterers, which will tend to lead to a central limit like Gaussian distribution. Moreover, since many scatterers are being included, correlations will build up and therefore ought to be included in the probability distribution of the slice.

So the evolution of the matrix with length (or whatever parameter he’s interested in) will be contained within A, B, and C themselves.

**1.1 Consequences of symmetries on A, B**

So these A’s and B’s must impose symmetry conditions on p(M) as well. Generic properties that the transfer matrix slice must satisfy are flux conservation and time-reversal invariance (perhaps). For instance, TRS requires:



which has the consequence that cross blocks are complex conjugates to each other. We can express this as Mkℓ = M\*k+N,ℓ+N. This can be enforced by making the coefficients multiplying the two matrix elements the same. So:



But it seems that we need a restriction on B as well. For instance suppose that p(M) were given by:



then we would require that A1 = A2 and B11 = B22 in order for <M1> = <M2>, or moreover that p(M1) = p(M2). Flux conservation requires:



I think in Mello’s paper/formulation, M was parameterized in terms of matrices that automatically satisfied flux conservation (at least in the WSL) and so there was no need for this ‘constraint’. If we combine this statement with the previous one, then we would have:



and this would require, so he says,



(real part is s = 1, imaginary part is s = 2). In any event we can make elements effectively equal by adjusting A and B appropriately.

**1.2 Modelling A and B in the various regimes**

Next he argues that pL(M) ansatz does in fact suffice, because these parameters Aμυ, and Bμυ;μ´υ´ can capture the ballistic, conducting, and insulating regimes. They would depend on disorder, length, and boundary conditions. Perhaps these parameters can be determined from a given model – like δ or Born. He claims that this formalism can in principle capture the transition b/c the number and strength of the B parameters will depend on dimension and disorder; higher dimension means more B’s because more channels, and changing disorder would mean channel interactions get weaker or stronger. Perhaps in localized regime, nearest neighbor channel interactions would prevail over longer ranged interactions. He proffers a model for A and B in the various regimes as follows:

i) ballistic: In this regime the transfer matrix ought to be almost diagaonal. So off-diagonal matrix elements should have probability 0 and diagonal ones should have probability centered about 1. And this could be modelled by Aμν = (ℓ/2L)δ­μν. And Bμν;μ´ν´ = (ℓ/L)δμνδμ´ν´. The L/ℓ dependence is enhancing the diagonal probabilities at the expence of the off diagonal ones I suppose, as L gets smaller. So p(M) would read:



ii) metallic: Here p(M) should be isotropic, and depend only on the transmission eigenvalues, or in other words on Tr(MM†) I believe. This can be modelled by Aμν = 0, Bμν;μ´ν´ = (ξ/L)δμν,μ´ν´.

Well anyway.

**2. Time development equation**

So now we need work out the FP equation of sorts,



So A, B, and C contain the length parameter. So ∂p/∂z will technically only act on A and B. But we’d prefer to transfer its action onto M so that we can write an evolution equation of the FP form. In particular it seems we want to write it so that we can say,



Let’s pretend for now that A and B are just single numbers. Then,



**2.1 Just take ∂/∂z and see what happens**

We don’t know how these depend on z. If we just take a derivative, we get:



Let’s say that C depends on z through A and B, which of course it does. So let’s work out,



and fill these in:



Seems we still need to know what Aʹ and Bʹ are. If we just imposed a diffusion law on the evolution equation, would that suffice to tell us what A(L) or B(L) were?

**2.2 Doing it his way,**

But, let’s see what he does. He manipulates it into a form that mimicks a total derivative in the parameter space. Define p0 = p/C. So let’s consider the two derivatives w/r to A, and B



and now the derivatives w/r to M are:



As can see, ∂/∂A contains an M, and ∂/∂B contains an M2. The presence of the M2 requires us to invoke ∂2/∂M2. Also, ∂/∂M and ∂2/∂M2 terms contain M-independent terms, and so it is inevitable that we will have on the RHS an extra constant in parameter space too. Seems we cannot get rid of this term by having an extra constant term on LHS because we’d have to multiple it by A’s, and B’s, which cannot be on that side. So let’s see how to get ∂/∂A and ∂/∂B in terms of ∂/∂M and ∂2/∂M2 – or vice versa rather.



These match his terms. Now observe:



and consider:



This one doesn’t have the necessary cancellations. But it looks like it’s just a sign issue. Even still, when I compute the following, it will not work out to a total derivative because that those γ constants won’t cancel, which is again, just a sign issue.



So I think things will cancel if signs defined properly. Anyway, he evidently gets



where,



For short, we write:



**2.3 OK so now what?**

Seems we’re comparing RHS to a geneal 1st order convection equation. Could say this is equal to ∂p/∂Y, if we were to say that fμμ = dB/dY and fμ = dA/dY. This would together determine how A and B evolve with this parameter Y. Anyway, we introduce two new parameters y1, y2, and write



He says he wishes to find the transformation which would make t1 = 1, t2 = 0, for instance. So we need,



Cruciallyish, we need to have equations for which y2, or more generally, yn≠1, can be set to a constant. Perhaps that is what motivates putting everything in surface PDE form. In principle, we should be able to solve these equations for y1 and y2.



He says sometimes y will end up being related to length. For example, consider A = 0, and B = 1/qL. Then,



So,



So for small lengths, y1 would equal L’ish.

**3. Evolution equations**

So first of all,

**3.1 Evolution equation for matrix**

So our evolution equation for M is:



which corresponds to a solution of form,



Let’s apply the operator,



OK whatever. So his general equation is:



We observe that this implies, being extra careful w/r to the real/imaginary parts,



But what to make of his extra constant term? I’m making the s, sʹ explicit here. Note these expectations pertain to all 4 sub-blocks of M, too. But how does this distinguish TRS from no TRS? Theory is that this FP equation here pertains to independent elements. And the Mʹ moments above, are really pertaining to independent elements. We know what the independent elements are in the β = 1, 2 cases, by symmetry. So for instance, current conservation requires:



whereas current conservation + TRS implies, additionally, that:



So in this way, the moment equations establish that certain elements will be equal to each other for cases of higher symmetry. This is like how we found in the ‘conserved quantities’ file in the Appendix that the probability distribution function evolves the way it does, regardless of what is conserved, but given the appropriate microscopic model

**3.2 Evolution equation for eigenvalues**

Now if we were to build a perturbative expansion off of Q, how would we do this? If evolving backwards, we’d say:



He seems to leave off the δδ part of δQ. Maybe I should see what he gets if he includes it. It looks like he is able to relate <δQ>Mʹ and <δQδQ>Mʹ back to the matrix Q itself. If I use my Δ model, then δM = M(1+Δ) – M = MΔ. So we’d have:



And so I wouldn’t be able to do this. Continuing,



Then we need the eigenvalues of this matrix. Calling the perturbation, V, these would be:



So he would find that all V terms can be related back to X itself. So all terms in the perturbation get put back in terms of the xj themselves. This is how he gets a closed expression for the evolution of xn, and consequently, the λn. Let’s just check our model again,



So evidently, we will not be able to relate the V’s purel back to X.

**3.2.1 Moments**

Now let’s look at the moments in the eigenvalue expansion. So first off, we have:



According to the perturbative expansion, the other moment we’ll need is δQmnδQmn\*. So let’s consider this:



But these results depend on the symmetries present. So let’s consider w/o TRS first, no implicit summation over α:



and then,



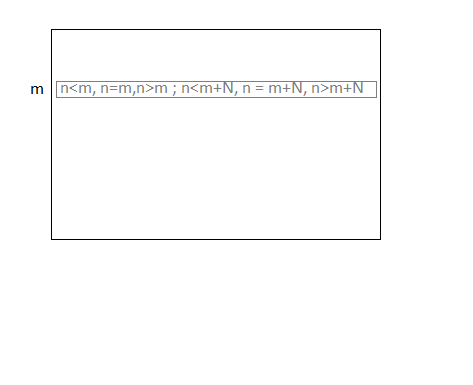
and,



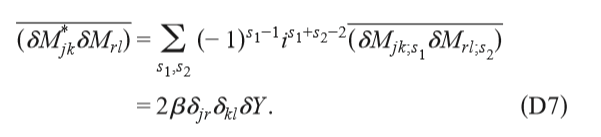
Well, I don’t want to do all this myself. So I’ll just quote results:



Should be thinking of it, ‘geometrically’, like this:



Going back to that first moment, I found that,



So it seems that we can say:



**3.2.2 So proceding with the evolution equation**

We got, explicitly including the other block



And so,



This is what he gets. And then the double moment would be:



I’m just quoting his result here, since he doesn’t explicitly give these correlations. So now we have a perturbative expansion of the x’s. And from this we can get a perturbative expansion of the λʹs. So,



Well, when I do it this way:



So that took a while, but finally got the same thing. OK so the λ moments are:



It appears he is leaving off the blue term, as aforementioned, and also the red term. And then the double moment is:



Now let’s work these out. First,



I’m just going to quote the results. First a definition:



and,



Now,



And so we can write this as:



Compare to:



If we were to define,



then we could write this as:



and so finally we have:



So we seem to have issues as the two middle terms are out of proportion, no? No, not in Q1D case where Kin = (1/2)Knn. So they are in proportion. And then if we take the small λn limit, it will reduce to the DMPK case. And the other one is:



as compared to:



And introducing the prior-defined k-matrix, we can write this as:



So this also has the appropriate proportion. So anyway, our evolution equation is (going to leave off the blue stuff – this would just change the function multiplying the extra γ term):



So we have:



So we get a remarkably similar form to our equation. Note that in this case, we get:



So could write as:



He says that when λ is small, then kii → 4, which is constant, and when you distribute the λi(1+λi) onto the γ term, it reduces back to the DMPK. I really think γ ought to be zero anyway. Whatever.



**Appendix**

So I’m gonna write down some properties of things he talks about.

**A.1 Properties of Q**

Let’s recall what M looks like for TRS:



where u and υ are N×N unitary matrices and λ is an N×N real diagonal matrix. The eigenvalue matrix λ can be expressed directly in terms of M. Forming Q = MM† and ζ = M†M, we have:



It follows as well that the eigenvalues of Q and ζ, which we’ll call xn, satisfy the relationships:



So he does perturbative expansions in terms of a matrix Q, instead, though.



Let’s diagonalize inner matrix. First the eigenvalues,



and what are the eigenvectors?



So eigenvectors/eigenvalues are:



So we can write Q as:



and say that the first (set of) eigenvalue(s) is:



Is the second set, their inverse?



So that checks out. And we can write it, as he says:



**A.2 Properties of ζ**

And now properties of **ζ**...



where u and υ are N×N unitary matrices and λ is an N×N real diagonal matrix. The eigenvalue matrix λ can be expressed directly in terms of M. Forming Q = MM† and ζ = M†M, we have:



It follows as well that the eigenvalues of Q and ζ, which we’ll call xn, satisfy the relationships:



So he does perturbative expansions in terms of a matrix Q, instead, though.



Let’s diagonalize inner matrix. Eigenvalues and eigenvectors are same, so we can write:



So we can write ζ as:



And so we can write it as:

